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Guaranteed Result for the Differential Game of Two Pursuers and One Evader on a Cylinder

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ABSTRACT

We study a pursuit differential game of two pursuers and one evader on a cylinder. We reduce the differential game to a differential game of two groups of countably many pursuers and one group of countably many evaders in $\mathbb{R}^2$ where all the players from each group are controlled by one control parameter subjected to geometric constraint. We obtain an estimate from above for the value of the game and construct strategies for the pursuers.

Keywords: Evader, optimal strategies, payoff functional, pursuer, value of game.

1. Introduction

The study about differential games was started by Isaacs (1965) and then developed by Friedman (1971), Krasovskii and Subbotin (1988), Pontryagin (1988), Petrosyan (1993), Hajek (1975) and others. Since then differential games of optimal approach were tremendously considered by many researchers (see, e.g., Pashkov and Terekhov (1987), Levchenkov and Pashkov (1990) and Ibragimov (2005)). Differential games of many pursuers pursuing single one were the attention of many researchers as well (see, e.g., Ramana and Kothari (2017) and Sun et al. (2017)).

Pashkov and Terekhov (1987) considered a differential game of approach involving one evader and two pursuers with a nonconvex payoff function. A generalized Bellman–Isaacs equation by Subbotin (1984) and Subbotin and Chentsov (1981) was used to find the value of the game involving one evader and one pursuer. Levchenkov and Pashkov (1990) examined the game of optimal approach of two inertial pursuers to a noninertial evader and constructed the value function for all possible positions of the game. The researchers decomposed the game space to find the (u, v)-stable functions composing the piecewise-smooth value function. The algorithms to find the value function were determined.

Ibragimov (1998) investigated a game of optimal pursuit in $\mathbb{R}^n$ where m pursuers pursue a single evader. The players perform simple motions with the controls of the first $k (k \leq m)$ pursuers subjected to integral constraints while the controls of the rest of the pursuers and the evader subjected to geometric constraints. In that paper, the value of the game was found and the optimal strategies of the players were constructed.

There are many works devoted to differential games in Hilbert spaces, for example, Satimov and Tukhtasinov (2006). The work Ibragimov (2005) studied a differential game of optimal pursuit with countably many pursuers and one evader in the space l_2 with elements $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k, ...)$,

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty,$$

and with inner product $(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k$. The optimal strategies for the players were constructed and the value of the game was found. In the works Ibragimov (2002, 2013) optimal strategies of pursuers were constructed for a differential game in a Hilbert space with integral constraints.

Differential games on the edges of regular polyhedrons were studied in the works Azamov et al. (2016, 2019) and necessary and sufficient conditions for the evasion from many pursuers were obtained. Moreover, the optimal numbers of pursuers to complete the game were found.

Some researchers worked on the games of approach on manifolds using the main equation of differential game theory (see, e.g., Kuchkarov (2007), Kuchkarov et al. (2012), Melikyan and Ovakimyan (1991) and Melikyan and Ovakimyan (1993)). Kuchkarov (2007) studied a differential game of optimal approach where players move in locally Euclidean spaces using Krasovskii's approach by Krasovskii and Subbotin (1974). The game involves countably many pursuers controlled by one parameter and one evader. The value of the game was obtained for any initial positions of players. Kuchkarov et al. (2012) investigated a pursuit problem of many pursuers when the evader moves on the surface of a cylinder. The researchers considered two cases where the pursuers move arbitrarily without phase constraints in the first case, and the pursuers move on the surface of the cylinder in the second case. For both the cases, necessary and sufficient conditions to complete pursuit were given. In the second case, it was shown that the pursuit differential game on the surface of cylinder is equivalent to a differential game on the plane with many groups of pursuers with the same control parameter.

In the present paper, we study a pursuit differential game of two pursuers and one evader on a cylinder. We reduce the differential game of two pursuers and one evader on the cylinder to a differential game of two groups of countably many pursuers, $..., x_i^{-1}, x_i^0, x_i^1, ... \in \mathbb{R}^2,\ i = 1, 2$, and a group of countably many evaders, $..., y^{-1}, y^0, y^1, ... \in \mathbb{R}^2$, respectively. We obtain an estimate for the value of the game and construct strategies for the pursuers.

2. Statement of Problem

Let

$$M = \left\{ x = \left(x_{(1)}, x_{(2)}, x_{(3)}\right) \in \mathbb{R}^3 \mid x_{(1)}^2 + x_{(2)}^2 = a^2,\ x_{(3)} \in \mathbb{R} \right\},$$

that is, M is the surface of a two-dimensional cylinder in $\mathbb{R}^3$ and $M_w, w \in M$, be the tangent plane to the surface of the cylinder M at point w. The motions of the pursuers x_i and the evader y on the surface of cylinder M are described by the following equations:

$$\begin{aligned} \dot{x}_i &= u_i, \quad x_i(t_0) = x_{i0}, \\ \dot{y} &= v, \qquad y(t_0) = y_0, \quad y_0 \neq x_{i0}, \end{aligned} \tag{1}$$

where $x_i, y \in M, u_i \in M_{x_i}, v \in M_y$, and u_i and v are control parameters of the players $x_i,\ i = 1, 2$, and y, respectively.

Definition 2.1. *A measurable function $u_i(t),\ t_0 \le t \le \theta,$ is called an admissible control of the pursuers x_i, if $|u_i(t)| \le \rho_i$ and along the solution $x_i(\cdot) = x_i(t)$ of the initial value problem $\dot{x}_i = u_i(t),\ x_i(t_0) = x_{i0}$, the inclusion*

$$u_i(t) \in M_{x_i(t)} \tag{2}$$

is satisfied, where $\rho_i,\ i = 1, 2,$ is a given positive number.

Definition 2.2. *A measurable function $v(t),\ t_0 \le t \le \theta,$ is called an admissible control of the evader y, if $|v(t)| \le \sigma$ and along the solution $y(\cdot) = y(t)$ of the initial value problem $\dot{y} = v(t),\ y(t_0) = y_0$, the inclusion*

$$v(t) \in M_{y(t)} \tag{3}$$

is satisfied, where σ is a given positive number.

Let $H(x, r)$ represents a circle with center at x and radius r on the tangent plane M_x.

Definition 2.3. *A function $U_i(t, x_i, y, v)$*

$$U_i : [t_0, \theta] \times M \times M \times H(y, \sigma) \to H(x_i, \rho_i),\ i = 1, 2,$$

is called strategy of pursuer x_i if for any admissible control $v(t),\ t_0 \le t \le \theta$ of the evader y, the initial value problem

$$\begin{aligned} \dot{x}_i &= U_i(x_i(t), y(t), v(t)), \quad x_i(t_0) = x_{i0}, \\ \dot{y} &= v(t), \quad y(t_0) = y_0, \end{aligned} \tag{4}$$

has a unique absolutely continuous solution $(x_i(t), y(t))$, $x_i(t), y(t) \in M$, and along the solution, the inclusion

$$U_i(x_i(t), y(t), v(t)) \in M_{x_i(t)} \tag{5}$$

holds.

The game is started at time t_0 and ended at the prescribed time θ. Let $d(x, y)$ be the distance between points $x, y \in M$ along the surface of cylinder M. The payoff functional is the distance between the evader and the closest to it pursuer at the time θ, that is $\min\limits_{i=1,2} d\left(x_i(\theta), y(\theta)\right)$.

In this paper, the payoff functional is the quantity

$$\gamma(x_1(\cdot), x_2(\cdot), y(\cdot)) = \min_{i=1,2} d(x_i(\theta), y(\theta)).$$

Problem. *Find an estimate for the value of the pursuit differential game and construct strategies for the pursuers.*

3. An Equivalent Game in the Plane

Let us reduce game (1)–(5) on the surface of the cylinder M to a specific game in $\mathbb{R}^2$ by unfolding M in $\mathbb{R}^2$ (Nikulin and Shafarevich (1983)). Such unfolding is presented by multivalued mapping which is inverse to the universal covering $\pi : \mathbb{R}^2 \to M$, that is local isometry. If $z \in M$, then the set of its preimages $\pi^{-1}(z)$ consists of the class of denumerable points, $..., z^{-1}, z_0, z_1, ... \in \mathbb{R}^2$ equivalent to each other. Now, the game (1)–(5) is reduced to a game in $\mathbb{R}^2$ in which two groups of countably many pursuers, $\pi^{-1}(x_i) = \{..., x_i^{-1}, x_i^0, x_i^1, ...\}$, $i = 1, 2$, pursue a group of countably many evaders, $\pi^{-1}(y) = \{..., y^{-1}, y^0, y^1, ...\}$. There exists a vector h such that (Nikulin and Shafarevich (1983))

$$x_i^{j+1} - x_i^j = h, \tag{6}$$

where $|h|$ is the length of circumference in the section of the cylinder M by a plane $x_3 = c$, and

$$d(x_i, y) = \min_{j,k} |x_i^j - y^k|, \;\; i = 1, 2. \tag{7}$$

The dynamics of the two groups of pursuers and the group of evaders are described by the following equations:

$$\begin{aligned} \dot{x}_i^j &= u_i, \quad x_i^j(t_0) = x_{i0}^j, \quad i = 1, 2, \\ \dot{y}^j &= v, \quad\;\; y^j(t_0) = y_0^j, \quad\; j = ..., -1, 0, 1, ..., \end{aligned} \tag{8}$$

where $x_i^j, u_i, y^j, v \in \mathbb{R}^2$, $x_{i0}^j \neq y_0^k$ for all $i = 1, 2$, $j, k \in \{..., -1, 0, 1, ...\}$, u_i and v are control parameters of the two groups of pursuers and the group of evaders, respectively, which satisfy the constraints

$$|u_i| \leq \rho_i, \quad i = 1, 2, \quad |v| \leq \sigma. \tag{9}$$

All players of each group are controlled by the same control parameter. Thus, the differential game on the cylinder M is reduced to a game of two groups of pursuers $\pi^{-1}(x_i), i = 1, 2$, and one group of evaders $\pi^{-1}(y)$ in the plane $\mathbb{R}^2$. The pursuers try to minimize the payoff functional $\min\limits_{i=1,2} d\,(x_i(\theta), y(\theta))$ and the evader tries to maximize it.

4. Main Result

We now formulate the main result of this paper. Consider the differential game (8)–(9). It can be shown that the attainability set of the pursuer x_i^j (respectively the evader y^j) from the initial position x_{i0}^j (respectively y_0^j) at time t_0 to the time θ is the circle $H(x_{i0}^j, \rho_i(\theta - t_0))$, $i = 1, 2$, (respectively $H(y_0^j, \sigma(\theta - t_0))$, $j = 1, 2, \ldots$.

Theorem 4.1. *Let $\rho_i \geq \sigma$, $i = 1, 2$, then the value of the game (8)–(9) is bounded above by the following number*

$$\gamma_0 = \min_{j,k} \gamma^{jk}, \quad j, k \in \{1, 2, \ldots\}, \tag{10}$$

where

$$\gamma^{jk} = \min \Big\{ l \geq 0 | \; H(y_0^0, \sigma(\theta - t_0)) \subset H(x_{10}^j, \rho_1(\theta - t_0) + l) \cup H(x_{20}^k, \rho_2(\theta - t_0) + l) \Big\}.$$

Proof. The proof of the theorem is based on two lemmas which we proved below. Consider a game involving one pursuer and one evader,

$$\begin{aligned} \dot{x} &= u, \quad x(t_0) = x_0, \quad |u| \leq \rho, \\ \dot{y} &= v, \quad y(t_0) = y_0, \quad |v| \leq \sigma. \end{aligned} \tag{11}$$

Let X be an n-dimensional half-space. Assume that $x_0, y_0 \in X$, and that the evader must move in X. There is no restriction of generality in assuming that $X = \{(\xi, \eta) | \; \eta \geq 0\}$. The payoff functional of the game is

$$\gamma(\theta) = |y(\theta) - x(\theta)|. \tag{12}$$

Let

$$\sigma(\theta - t) \geq y_2, \quad \frac{y_2}{\sigma(\theta - t)} \leq \frac{x_2}{R}, \tag{13}$$

where

$$R = \left(\left(\sqrt{\sigma^2(\theta - t)^2 - y_2^2} + |y_1 - x_1| \right)^2 + x_2^2 \right)^{1/2}.$$

Then we set

$$\gamma(t, x_1, x_2, y_1, y_2) = R - \rho(\theta - t). \tag{14}$$

If at least one of the inequalities (13) is not satisfied, then we set

$$\gamma(t, x_1, x_2, y_1, y_2) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} - (\rho - \sigma)(\theta - t). \tag{15}$$

We illustrate inequalities (13). The inequality $\sigma(\theta - t) \geq y_2$ shows that the circumference $S(y, \sigma(\theta - t))$ intersects ξ-axis at points $(y_1 + r, 0)$ and $(y_1 - r, 0)$, where $r = \sqrt{\sigma^2(\theta - t)^2 - y_2^2}$. The second inequality in (13) shows that the point $y = (y_1, y_2)$ is either in the triangle with vertices $x = (x_1, x_2)$, $(x_1, 0)$ and $(x_1 + |y_1 - x_1| + r, 0)$ or in the triangle with vertices $x = (x_1, x_2)$, $(x_1, 0)$ and $(x_1 - |y_1 - x_1| - r, 0)$
(see Figures 1 and 2).

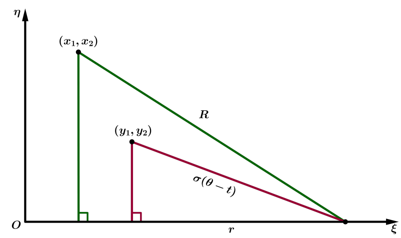

Figure 1: Positions of pursuer and evader in the half plane when $x_1 < y_1$

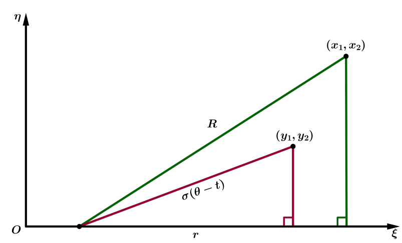

Figure 2: Positions of pursuer and evader in the half plane when $x_1 > y_1$

Note that not for all points of these triangles, inequalities in (13) are satisfied.

Lemma 4.1. *If $\rho > \sigma$, then the function γ defined by the equations (14) and (15) is the value function of game (11)–(12).*

Proof. The proof of the lemma is as follows.

Definition 4.1. *The strategies U_{i0} of the pursuers x_i are said to be optimal if*

$$\inf_{U_1, U_2} \Gamma(U_1, U_2) = \Gamma(U_{10}, U_{20}), \;\; \Gamma(U_1, U_2) = \sup_{v(\cdot)} \min_{i=1,2} d\left(x_i(\theta), y(\theta)\right),$$

where U_i are strategies of the pursuers $x_i, i = 1,2$ respectively, and $v(\cdot)$ is an admissible control of the evader y.

Definition 4.2. *If the following limit*

$$\frac{\partial w(t,x,y)}{\partial(1,u,v)} = \lim_{\delta\to 0} \frac{w(t+\delta, x+\delta u, y+\delta v) - w(t,x,y)}{\delta}$$

exists, it is called the directional derivative of the function $w(t,x,y)$ in the direction $(1,u,v)$.

Definition 4.3. *A function $f : A \to \mathbb{R}^n,\ A \subset \mathbb{R}^n$, is locally Lipschitz at $x_0 \in A$ if there exists a constant K such that $|f(x) - f(x_0)| \le K|x - x_0|$.*

Prop 4.1. *A locally Lipschitz function $w(t,x,y),\ w : [t_0,\theta] \times \mathbb{R}^n \times \mathbb{R}^n$, that is differentiable in any direction $(1,l) \in \mathbb{R}^{2n+1}$ is the potential of differential game (11), (12) if and only if, for all $(t,x,y) \in [t_0,\theta) \times \mathbb{R}^n \times \mathbb{R}^n$, it satisfies the boundary condition*

$$w(\theta,x,y) = \gamma(\theta,x,y) \tag{16}$$

and the inequalities

$$\max_{|v|\le\sigma} \min_{|u|\le\rho} \frac{\partial w(t,x,y)}{\partial(1,u,v)} \le 0, \tag{17}$$

$$\min_{|u|\le\rho} \max_{|v|\le\sigma} \frac{\partial w(t,x,y)}{\partial(1,u,v)} \ge 0. \tag{18}$$

We say that the function w satisfies the condition of u-stability (respectively v-stability) if inequality (17) (respectively (18)) is satisfied.
If the function w is differentiable at a point (t,x,y), then (17) and (18) take the form

$$\frac{\partial w(t,x,y)}{\partial t} + \min_{|u|\le\rho} \frac{\partial w(t,x,y)}{\partial x} u + \max_{|v|\le\sigma} \frac{\partial w(t,x,y)}{\partial y} v = 0. \tag{19}$$

Let (13) be satisfied. Without any loss of generality assume that $x_1 \le y_1$. Consider two cases, $x_1 < y_1$ and $x_1 = y_1$.
Let $x_1 < y_1$. Define the strategy of the pursuer as follows:

$$U(x,y) = (\rho\sin\alpha, -\rho\cos\alpha),\ \ \alpha = \arccos\left(\frac{x_2}{R}\right). \tag{20}$$

The function (14) is differentiable and due to (19), the condition of u-stability takes the form

$$\gamma_t + \gamma_{x_1} U_1 + \gamma_{x_2} U_2 + \gamma_{y_1} v_1 + \gamma_{y_2} v_2 \le 0 \tag{21}$$

for all $v, |v| \le \sigma$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \gamma_{y_1} v_1 + \gamma_{y_2} v_2 &\le \sqrt{\gamma_{y_1}^2 + \gamma_{y_2}^2}\sqrt{v_1^2 + v_2^2} \\ &\le \sqrt{\frac{(y_1 - x_1 + r)^2}{R^2} + \frac{y_2^2 (y_1 - x_1 + r)^2}{R^2 r^2}} \cdot \sigma \\ &= \frac{\sigma^2(\theta - t)(y_1 - x_1 + r)}{Rr}. \end{aligned}$$

Now using this and equation (20), we estimate the left hand side of (21) as follows:

$$\begin{aligned} & \gamma_t + \gamma_{x_1} U_1 + \gamma_{x_2} U_2 + \gamma_{y_1} v_1 + \gamma_{y_2} v_2 \\ \le \quad & \frac{-\sigma^2(\theta - t)(y_1 - x_1 + r)}{Rr} + \rho - \frac{(y_1 - x_1 + r)}{R}\left(\rho \cdot \frac{(y_1 - x_1 + r)}{R}\right) \\ & + \frac{x_2}{R}\left(-\rho \cdot \frac{x_2}{R}\right) + \frac{\sigma^2(\theta - t)(y_1 - x_1 + r)}{Rr} \\ = \quad & \frac{-\sigma^2(\theta - t)(y_1 - x_1 + r)}{Rr} + \rho - \frac{\rho}{R^2}\left((y_1 - x_1 + r)^2 + x_2^2\right) \\ & + \frac{\sigma^2(\theta - t)(y_1 - x_1 + r)}{Rr} \quad = \quad 0. \end{aligned}$$

Thus, the function $\gamma(t, x_1, x_2, y_1, y_2)$ satisfies the condition of u-stability. Next, let $x_1 = y_1$ (see Figure 3).

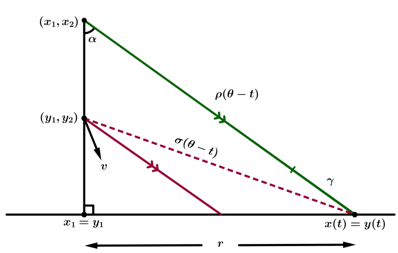

Figure 3: The case when $x_1 = y_1$

Define the strategy of the pursuer as follows:

$$U(x, y) = \begin{cases} (\rho \sin\alpha, -\rho\cos\alpha), & |v_1| > \rho\sin\alpha, \\ \left(v_1, -\sqrt{\rho^2 - v_1^2}\right), & |v_1| \le \rho\sin\alpha. \end{cases} \tag{22}$$

For definitness, assume that $v_1 \geq 0$. Let $v_1 > \rho \sin\alpha$.
Similar to the case when $x_1 < y_1$, it can be concluded that the function $\gamma(t, x_1, x_2, y_1, y_2)$ satisfies the condition of u-stability.
Let now $0 \leq v_1 \leq \rho \sin\alpha = \dfrac{r\rho}{R}$. Since $x_1 = y_1$ and $U_1 = v_1$, we have

$$\begin{array}{ll} r = r(t) = \sqrt{\sigma^2(\theta-t)^2 - y_2^2}, & r(t,\delta) = \sqrt{\sigma^2(\theta-t-\delta)^2 - (y_2+v_2\delta)^2}, \\ R = R(t) = \sqrt{r^2 + x_2^2}, & R(t,\delta) = \sqrt{r^2(t,\delta) + (x_2+U_2\delta)^2}, \\ \gamma(t) = R - \rho(\theta - t), & \gamma(t,\delta) = R(t,\delta) - \rho(\theta-t-\delta). \end{array}$$

Since $|v_2| = \sqrt{|v|^2 - v_1^2} \leq \sqrt{\sigma^2 - v_1^2}$, therefore
$r(t,\delta) \leq \sqrt{\sigma^2(\theta-t-\delta)^2 - \left(y_2 - \delta\sqrt{\sigma^2 - v_1^2}\right)^2} = \bar{r}(\delta)$.
By (17), the function γ satisfies the condition of u-stability when

$$\lim_{\delta\to 0^+} \frac{\gamma(t+\delta, x_1+U_1\delta, x_2+U_2\delta, y_1+v_1\delta, y_2+v_2\delta) - \gamma(t, x_1, x_2, y_1, y_2)}{\delta} \leq 0 \tag{23}$$

for all v, $|v| \leq \sigma$. Equivalently,

$$\lim_{\delta\to 0^+} \frac{R(t,\delta) - \rho(\theta-t-\delta) - (R - \rho(\theta-t))}{\delta} \leq 0.$$

It suffices to show that

$$\rho + \lim_{\delta\to 0^+} \frac{\sqrt{\bar{r}^2(\delta) + (x_2 - \delta\sqrt{\rho^2 - v_1^2})^2} - \sqrt{r^2 + x_2^2}}{\delta} \leq 0, \quad 0 \leq v_1 \leq \frac{r\rho}{R},$$

which is equivalent to

$$R\rho - \sigma^2(\theta-t) + y_2\sqrt{\sigma^2 - v_1^2} - x_2\sqrt{\rho^2 - v_1^2} \leq 0. \tag{24}$$

To find the maximum of the left hand side of (24) on the interval $0 \leq v_1 \leq \dfrac{r\rho}{R}$, consider the function

$$f(\xi) = y_2\sqrt{\sigma^2 - \xi} - x_2\sqrt{\rho^2 - \xi}, \quad 0 \leq \xi \leq \left(\frac{\rho r}{R}\right)^2.$$

The equation

$$f'(\xi) = -\frac{y_2}{2\sqrt{\sigma^2 - \xi}} + \frac{x_2}{2\sqrt{\rho^2 - \xi}} = 0$$

has the root $\xi_0 = \dfrac{x_2^2\sigma^2 - y_2^2\rho^2}{x_2^2 - y_2^2}$, which is positive since $\dfrac{y_2}{\sigma(\theta-t)} \leq \dfrac{x_2}{R} < \dfrac{x_2}{\rho(\theta-t)}$, that is, $y_2\rho \leq x_2\sigma$.

It is clear that $f(\xi)$ increases on $[0, \xi_0]$. Show that $\xi_0 \geq (\rho \sin\alpha)^2 = \left(\frac{\rho r}{R}\right)^2$. Indeed, the inequality

$$\frac{x_2^2\sigma^2 - y_2^2\rho^2}{x_2^2 - y_2^2} \quad \geq \quad \frac{\rho^2 r^2}{R^2},$$

is equivalent to

$$r^2x_2^2\sigma^2 - r^2y_2^2\rho^2 + x_2^4\sigma^2 - x_2^2y_2^2\rho^2 \quad \geq \quad r^2x_2^2\rho^2 - r^2y_2^2\rho^2,$$

which can be written as

$$\sigma^2(r^2 + x_2^2) \quad \geq \quad \rho^2(y_2^2 + r^2),$$

followed by

$$x_2^2 + r^2 \geq \rho^2(\theta - t)^2,$$

results in the following obvious inequality

$$R^2 \geq \rho^2(\theta - t)^2.$$

Thus, $f(\xi)$ increases on $[0, \rho\sin\alpha]$ as well.
Hence, the left hand side of (24) takes its maximum at $v_{10} = \rho\sin\alpha = \frac{\rho r}{R}$. Therefore, to show (24), we show that

$$R\rho - \sigma^2(\theta - t) + y_2\sqrt{\sigma^2 - \frac{\rho^2r^2}{R^2}} - x_2\sqrt{\rho^2 - \frac{\rho^2r^2}{R^2}} \leq 0.$$

Since $R^2 = r^2 + x_2^2$ and $v_{10} = \frac{\rho r}{R}$, we have

$$\left(R - \frac{x_2^2}{R}\right)\rho + y_2\sqrt{\sigma^2 - v_{10}^2} \quad \leq \quad \sigma^2(\theta - t),$$

or the same

$$rv_{10} + y_2\sqrt{\sigma^2 - v_{10}^2} \quad \leq \quad \sigma^2(\theta - t). \tag{25}$$

To see this, we apply the inequality

$$a_1b_1 + a_2b_2 \leq \sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2}$$

to the left hand side of (25). Indeed,

$$\begin{aligned} rv_{10} + y_2\sqrt{\sigma^2 - v_{10}^2} &\le \sqrt{r^2 + y_2^2}\cdot\sqrt{v_{10}^2 + \sigma^2 - v_{10}^2} \\ &= \sqrt{\sigma^2(\theta - t)^2}\cdot\sqrt{\sigma^2} = \sigma^2(\theta - t). \end{aligned}$$

Thus, the function $\gamma(t, x_1, x_2, y_1, y_2)$ satisfies inequality (23) and hence, it is u-stable.
Let now (13) is not satisfied. Then the function (15) is differentiable and it satisfies the condition of u-stability in the plane (Krasovskii and Subbotin (1974)). The proof of the lemma is completed. □

Consider now a differential game of two pursuers and one evader,

$$\begin{aligned} \dot{x}_i &= u_i, \quad x_i(t_0) = x_{i0}, \quad |u_i| \le \rho_i, \quad i = 1, 2, \\ \dot{y} &= v, \qquad y(t_0) = y_0, \qquad |v| \le \sigma. \end{aligned} \tag{26}$$

The duration of the game, θ is fixed. The payoff functional of the game is

$$\gamma(\theta) = \min_{i=1,2} |y(\theta) - x_i(\theta)|, \tag{27}$$

that is $\gamma(\theta)$ is the distance between the evader and the closest to it pursuer at the time θ.

Lemma 4.2. *Let $\rho_i \ge \sigma,\ i = 1, 2$. Then*

$$\gamma^* = \min\left\{ l \ge 0 |\ H(y_0, \sigma(\theta - t_0)) \subset \bigcup_{i=1}^{2} H(x_{i0}, \rho_i(\theta - t_0) + l) \right\} \tag{28}$$

is the value of game (26)–(27).

Proof. Note that if $\gamma^* = 0$, then (28) implies that
$H(y_0, \sigma(\theta - t_0)) \subset H(x_{10}, \rho_1(\theta - t_0)) \cup H(x_{20}, \rho_2(\theta - t_0))$ and conclusion of Lemma 4.2 is that there exist strategies of pursuers such that for any behaviour of evader, $x_1(\theta) = y(\theta)$ or $x_2(\theta) = y(\theta)$.

In the case where $\gamma^* = 0$, without loss of generality we assume that $\theta = \theta_0$, where $\theta_0 = \min\left\{ t \ge t_0 |\ H(y_0, \sigma(t - t_0)) \subset \bigcup_{i=1}^{2} H(x_{i0}, \rho_i(t - t_0)) \right\}$.

Because if, for the game (26) with payoff functional $\gamma(\theta_0)$, the value of the game is $\gamma^* = 0$, then for the game (26)–(27), the value is $\gamma^* = 0$ as well.

Indeed, if $\theta_0 \leq \theta$, and $x_{i_0}(\theta_0) = y(\theta_0)$ for some $i_0 \in \{1, 2\}$, then the pursuers using controls $u_1(t) = v(t),\ u_2(t) = v(t),\ \theta_0 < t \leq \theta$, guarantee $x_{i_0}(\theta) = y(\theta)$.

Let $\gamma^* > 0$. Consider two cases:
1: $H(y_0, \sigma(\theta - t_0)) \subset H(x_{k0}, \rho_k(\theta - t_0) + \gamma^*)$ for some $k \in \{1, 2\}$.
2: $H(y_0, \sigma(\theta - t_0)) \not\subset H(x_{i0}, \rho_i(\theta - t_0) + \gamma^*),\ i = 1, 2$, but
$H(y_0, \sigma(\theta - t_0)) \subset \bigcup_{i=1}^{2} H(x_{i0}, \rho_i(\theta - t_0) + \gamma^*)$.

In Case 1, the differential game is reduced to the game of one pursuer x_k and one evader, and γ^* in (15) is the value of the game. In Case 2, one can show that
$S_0 = S(y_0, \sigma(\theta - t_0)) \cap S(x_{10}, \rho_1(\theta - t_0) + \gamma^*) \cap S(x_{20}, \rho_2(\theta - t_0) + \gamma^*) \neq \emptyset$.

Let $M \in S_0$. Pass a perpendicular line p to segment $[x_{10}, x_{20}]$ from the point M. The line p divides the plane into two half-planes, X^- and X^+ (see Figure 4).

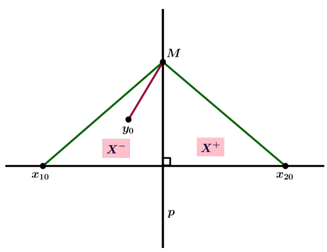

Figure 4: Position of the evader inside the triangle $x_{10}x_{20}M$

Without loss of generality assume that the line p coincides with y-axis. The attainability set of the pursuers $x_i,\ i = 1, 2$, from the point x_{i0} at $t = t_0$ to the time θ is $S(x_{i0}, \rho_i(\theta - t_0))$ and that of the evader y from the point y_0 at $t = t_0$ to the time θ is $S(y_0, \sigma(\theta - t_0))$. We introduce a fictitious evader whose motion is described by the following equation:

$$\dot{\bar{y}} = \bar{v}, \quad \bar{y}(t_0) = \bar{y}_0, \quad |\bar{v}| \leq \sigma, \tag{29}$$

where $\bar{y}, \bar{v} \in \mathbb{R}^2$ and $\bar{y}_0 = (-y_{10}, y_{20})$. The attainability set of the fictitious evader from the point $\bar{y}_0$ at $t = t_0$ to the time θ is $S(\bar{y}_0, \sigma(\theta - t_0))$. Let the controls of the evader and the fictitious evader be

$$v(t) = (v_1(t), v_2(t)) \tag{30}$$

and

$$\bar{v}(t) = (-\bar{v}_1(t), \bar{v}_2(t)) \tag{31}$$

respectively. It is not difficult to show that $y(t)$ and $\bar{y}(t)$ are symmetric with respect to y-axis. Therefore, if $y(t) \in X^-$, then $\bar{y}(t) \in X^+$ or if $y(t) \in X^+$, then $\bar{y}(t) \in X^-$. Hence, for each time $t \geq 0$, each of half planes X^- and X^+ contains one of the evaders $y(t)$ and $\bar{y}(t)$. Let the pursuer x_1 pursues the evader (real or fictitious) in the half plane X^-, and let the pursuer x_2 pursues the evader in the half plane X^+. Then by Lemma 4.1 there are strategies of pursuers x_1 and x_2 such that at θ the distance between pursuer x_1 and the evader in X^-, and the distance between pursuer x_2 and the evader X^+ is less than or equal to γ. However, one of these evaders is real evader. Hence, if $y(\theta) \in X^-$, then $|x_1(\theta) - y(\theta)| \leq \gamma$, and if $y(\theta) \in X^+$, then $|x_2(\theta) - y(\theta)| \leq \gamma$. The proof of the lemma is completed. □

Finally, we prove Theorem 4.1. Let

$$H(y_0^0, \sigma(\theta - t_0)) \subset H(x_{10}^{j_0}, \rho_1(\theta - t_0) + \gamma_0) \cup \; H(x_{20}^{k_0}, \rho_2(\theta - t_0) + \gamma_0) \tag{32}$$

for some j_0 and k_0. To prove the theorem one has to show that for any r there exist numbers j and k such that at least one of the following inequalities is satisfied

$$|x_1^j(\theta) - y^r(\theta)| \leq \gamma_0, \quad |x_2^k(\theta) - y^r(\theta)| \leq \gamma_0.$$

It sufficient to consider the game only with one evader $y^0(t)$ with initial position y_0^0 since if $|x_i^j(\theta) - y^0(\theta)| \leq \gamma_0$ for some i and j, then by (6) we have for any r, that $x_{i0}^{j+r} = x_{i0}^j + rh$ and $y_0^r = y_0^0 + rh$, and so

$$|x_i^{j+r}(\theta) - y^r(\theta)| = |x_i^j(\theta) - y^0(\theta)| \leq \gamma_0.$$

In other words, if the pursuer x_i^j is in γ_0 vicinity of evader y^0 at θ, then the pursuer x_i^{j+r} will be in γ_0 vicinity of evader y^r at θ. Then by Lemma 4.2 equation (32) allows us to state that γ_0 is guaranteed for the pursuers $x_1^{j_0}$ and $x_2^{k_0}$ meaning that at least one of the following inequalities is true:

$$|x_1^{j_0}(\theta) - y^0(\theta)| \leq \gamma_0, \quad |x_1^{k_0}(\theta) - y^0(\theta)| \leq \gamma_0.$$

This completes the proof of Theorem 4.1. □

5. Conclusion

In this paper, we studied a pursuit differential game of two pursuers and one evader on a cylinder. We reduced the game to an equivalent differential game of two groups of countably many pursuers and one group of countably many evaders in $\mathbb{R}^2$ by unfolding the cylinder. To obtain the main result of this paper, we studied differential games involving one pursuer and one evader as well as two purseurs and one evader in $\mathbb{R}^2$. We obtained an estimate for the value of the game and constructed strategies for the pursuers.

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